

A Leray spectral sequence for noncommutative differential fibrations

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Abstract

This paper describes the Leray spectral sequence associated to a differential fibration. The differential fibration is described by base and total differential graded algebras. The cohomology used is non-commutative differential sheaf cohomology. For this purpose, a sheaf over an algebra is a left module with zero curvature covariant derivative. As a special case, we can recover the Serre spectral sequence for a noncommutative fibration.

1 Introduction

This paper uses the idea of noncommutative sheaf theory introduced in [2]. This is a differential definition, so the algebras involved have to have a differential structure. Essentially having zero derivative is used to denote ‘locally constant’, which is a term of uncertain meaning for an algebra. Working rather vaguely, one might think of considering the total space of a sheaf over a manifold as locally inheriting the differential structure of the manifold, via the homeomorphism between a neighbourhood of a point in the total space and an open set in the base space. This allows us to lift a vector at a point of the base space to a unique vector at every point of the preimage of that point in the total space. This lifting should allow us to give a covariant derivative on the functions on the total space. Further, the local homeomorphisms suggest that the resulting covariant derivative has zero curvature. In [2] it is shown that a zero curvature covariant derivative on a module really does allow us to reproduce some of the main results of sheaf

cohomology. In this paper we shall consider another of the main results of sheaf cohomology, the Leray spectral sequence.

Ideally it would be nice to have a definition which did not involve differential structures, but there are several comments to be made on this: When Connes calculated the cyclic cohomology of the noncommutative torus [8], he used a subalgebra of rapidly decreasing sequences, effectively placing differential methods at the heart of noncommutative cohomology. It is not obvious what a *calculable* purely algebraic (probably read C^* algebraic) sheaf cohomology theory would be – though maybe the theory of quantales [14] might give a clue. Secondly, even if there were a non-differential definition, it would likely be complementary to the differential definition. The relation between de Rham and topological cohomology theories is fundamental to a lot of mathematics, it would make no sense to delete either. Finally, in mathematics today, differential graded algebras arising from several constructions are considered interesting objects in their own right, and many applications to Physics are phrased in terms of differential forms or vector fields.

There are four main motivations behind this paper. One is that the Leray spectral sequence seems a natural continuation from the sheaf theory and Serre spectral sequence in [2]. Another is a step in finding an analogue of the Borel-Weil-Bott theorem for representations of quantum groups (see [4]). One motivation we should look at in more detail is contained in the papers [9, 10]. These papers are about noncommutative fibrations. The differences in approach can be summarised in two sentences: We require that the algebras have differential structures, and [9, 10] do not. The papers [9, 10] require that the base is commutative, and we do not. One interesting point is that the method of [10] makes use of the classical Leray spectral sequence of a fibration with base a simplicial complex. The fourth motivation is noncommutative algebraic topology, where we would define a fibration on a category whose objects were differential graded algebras. The interesting question is then whether there is a corresponding idea of cofibration in the sense of model categories [15].

The example of the noncommutative Hopf fibration in [2] shows that a differential fibration need not have a commutative base. The example in Section 6 was made by taking a differential picture of a fibration given as an example in [9] (the base is the functions on the circle), and so it can be considered a noncommutative fibration in both senses. It would be useful to consider whether higher dimensional constructions, such as the 4-dimensional orthogonal quantum sphere in [1], also give examples of differential fibrations. As differential calculi on finite groups are quite well

understood (e.g. see [7, 11]), it would be interesting to ask what a differential fibration corresponds to in this context. From the point of view of methods in mathematical Physics, the quantisation of twistor theory (see [5]) is likely to provide some examples.

This paper is based on part of the content of the Ph.D. thesis [12].

2 Spectral sequences

This is standard material, and we use [13] as a reference. We will give quite general definitions, but likely not the most general possible.

2.1 What is a spectral sequence?

A spectral sequence consists of series of pages (indexed by r) and objects $\mathcal{E}_r^{p,q}$ (e.g. vector spaces), where r, p, q are integers. We take $r \geq 1$ and $p, q \geq 0$, and set $\mathcal{E}_r^{p,q} = 0$ if $p < 0$ or $q < 0$. There is a differential

$$d_r : \mathcal{E}_r^{p,q} \longrightarrow \mathcal{E}_r^{p+r, q+1-r}$$

such that $d_r d_r = 0$. As $d_r d_r = 0$, we can take a quotient (in our case, quotient of vector spaces)

$$\frac{\ker d_r : \mathcal{E}_r^{p,q} \rightarrow \mathcal{E}_r^{p+r, q+1-r}}{\operatorname{im} d_r : \mathcal{E}_r^{p-r, q+r-1} \rightarrow \mathcal{E}_r^{p,q}} = H_r^{p,q}$$

Then the rule for going from page r to page $r+1$ is $\mathcal{E}_{r+1}^{p,q} = H_r^{p,q}$. The maps d_{r+1} are given by a detailed formula on $H_r^{p,q}$. The idea is that eventually the $\mathcal{E}_r^{p,q}$ will become fixed for r large enough. The spectral sequence is said to converge to these limiting cases $\mathcal{E}_\infty^{p,q}$ as r increases.

2.2 The spectral sequence of filtration

A decreasing filtration of a vector space V is a sequence of subspaces $F^m V$ ($m \in \mathbb{N}$) for which $F^{m+1} V \subset F^m V$. The reader should refer to [13] for the details of the homological algebra used to construct the spectral sequence. We will merely quote the results.

Remark 2.1 *Start with a differential graded module C^n (for $n \geq 0$) and $d : C^n \rightarrow C^{n+1}$ with $d^2 = 0$. Suppose that C has a filtration $F^m C \subset C = \bigoplus_{n \geq 0} C^n$ for $m \geq 0$ so that:*

(1) $dF^m C \subset F^m C$ for all $m \geq 0$ (i.e. the filtration is preserved by d);

- (2) $F^{m+1}C \subset F^mC$ for all $m \geq 0$ (i.e. the filtration is decreasing);
(3) $F^0C = C$ and $F^mC^n = F^mC \cap C^n = \{0\}$ for all $m > n$ (a boundedness condition).

Then there is a spectral sequence $(\mathcal{E}_r^{*,*}, d_r)$ for $r \geq 1$ (r counts the page of the spectral sequence) with d_r of bidegree $(r, 1-r)$ and

$$\begin{aligned} \mathcal{E}_1^{p,q} &= H^{p+q}(F^pC/F^{p+1}C) \\ &= \frac{\ker d : F^pC^{p+q}/F^{p+1}C^{p+q} \rightarrow F^pC^{p+q+1}/F^{p+1}C^{p+q+1}}{\operatorname{im} d : F^pC^{p+q-1}/F^{p+1}C^{p+q-1} \rightarrow F^pC^{p+q}/F^{p+1}C^{p+q}} . \end{aligned} \quad (1)$$

In more detail, we define

$$\begin{aligned} Z_r^{p,q} &= F^pC^{p+q} \cap d^{-1}(F^{p+r}C^{p+q+1}) , \\ B_r^{p,q} &= F^pC^{p+q} \cap d(F^{p-r}C^{p+q-1}) , \\ \mathcal{E}_r^{p,q} &= Z_r^{p,q} / (Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}) . \end{aligned}$$

The differential $d_r : \mathcal{E}_r^{p,q} \rightarrow \mathcal{E}_r^{p+r,q-r+1}$ is the map induced on quotienting $d : Z_r^{p,q} \rightarrow Z_r^{p+r,q-r+1}$. The diligent reader should remember an important point here, when reading the seemingly innumerable differentials in the pages to come. There is really only one differential d – its domain or codomain may be different subspaces with different quotients applied, but the same d lies behind them all.

The spectral sequence converges to $H^*(C, d)$ in the sense that

$$\mathcal{E}_\infty^{p,q} \cong \frac{F^p H^{p+q}(C, d)}{F^{p+1} H^{p+q}(C, d)} ,$$

where $F^p H^*(C, d)$ is the image of the map $H^*(F^pC, d) \rightarrow H^*(C, d)$ induced by inclusion $F^pC \rightarrow C$.

2.3 The classical Leray spectral sequence

The statement of the general Leray spectral sequence can be found in [6]. We shall omit the supports and the subsets as we are only currently interested in a non commutative analogue of the spectral sequence. Then the statement reads that, given $f : X \rightarrow Y$ and \mathcal{S} a sheaf on X , that there is a spectral sequence

$$E_2^{pq} = H^p(Y, H^q(f, f|\mathcal{S}))$$

converging to $H^{p+q}(X, \mathcal{S})$. Here $H^q(f, f|\mathcal{S})$ is a sheaf on Y which is given by the presheaf for an open $U \subset Y$

$$U \mapsto H^q(f^{-1}U; \mathcal{S}|_{f^{-1}U}).$$

Here $f^{-1}U$ is an open set of X , and $\mathcal{S}|_{f^{-1}U}$ is the sheaf \mathcal{S} restricted to this open set.

We shall consider the special case of a differential fibration. This is the background to the Serre spectral sequence, but we consider a sheaf on the total space. The Leray spectral sequence of a fibration is a spectral sequence whose input is the cohomology of the base space B with coefficients in the cohomology of the fiber F , and converges to the cohomology of the total space E . Here

$$\pi : E \rightarrow B$$

is a fibration with fiber F . The difference of this from the Serre spectral sequence is that the cohomology may have coefficients in a sheaf on E .

3 Noncommutative differential calculi and sheaf theory

Take a possibly noncommutative algebra A . Then a differential calculus (Ω^*A, d) is given by the following.

Definition 3.1 *A differential calculus (Ω^*A, d) on A consists of vector spaces $\Omega^n A$ with operators \wedge and d so that*

- 1) $\wedge : \Omega^r A \otimes \Omega^m A \longrightarrow \Omega^{r+m} A$ is associative (we do not assume any graded commutative property)
- 2) $\Omega^0 A = A$
- 3) $d : \Omega^n A \rightarrow \Omega^{n+1} A$ with $d^2 = 0$
- 4) $d(\xi \wedge \eta) = d\xi \wedge \eta + (-1)^r \xi \wedge d\eta$ for $\xi \in \Omega^r A$
- 5) $\Omega^1 A \wedge \Omega^n A = \Omega^{n+1} A$.
- 6) $A.dA = \Omega^1 A$

Note that many differential graded algebras do not obey (5), but those in classical differential geometry do, and it will be true in all our examples. There is only one place where we require (5), and we will point it out at the time.

A special case of \wedge shows that each $\Omega^n A$ is an A -bimodule. We will often use $|\xi|$ for the degree of ξ , if $\xi \in \Omega^n A$, then $|\xi| = n$.

In the differential graded $(\Omega^n A, \wedge, d)$, we have $d^2 = 0$. This means that

$$\text{im } d : \Omega^{n-1} A \longrightarrow \Omega^n A \subset \ker d : \Omega^n A \longrightarrow \Omega^{n+1} A .$$

Then we define the de Rham cohomology as

$$H_{\text{dR}}^n(A) = \frac{\ker d : \Omega^n A \longrightarrow \Omega^{n+1} A}{\text{im } d : \Omega^{n-1} A \longrightarrow \Omega^n A} .$$

We give the usual idea of covariant derivatives on left A modules by using the left Liebnitz rule:

Definition 3.2 *Given a left A -module E , a left A -covariant derivative is a map $\nabla : E \rightarrow \Omega^1 A \otimes_A E$ which obeys the condition $\nabla(a.e) = da \otimes e + a.\nabla e$ for all $e \in E$ and $a \in A$.*

After the fashion of the de-Rham complex, we can attempt to extend the covariant derivative to a complex as follows:

Definition 3.3 *[2] Given (E, ∇) a left A -module with covariant derivative, define*

$$\nabla^{[n]} : \Omega^n A \otimes_A E \rightarrow \Omega^{n+1} A \otimes_A E, \quad \omega \otimes e \mapsto d\omega \otimes e + (-1)^n \omega \wedge \nabla e.$$

Then the curvature is defined as $R = \nabla^{[1]} \nabla : E \rightarrow \Omega^2 A \otimes E$, and is a left A -module map. The covariant derivative is called flat if the curvature is zero.

However, the curvature forms an obstruction to setting up a cohomology, as we now show:

Proposition 3.4 *[2] For all $n \geq 0$, $\nabla^{[n+1]} \circ \nabla^{[n]} = \text{id} \wedge R : \Omega^n A \otimes_A E \rightarrow \Omega^{n+2} A \otimes_A E$.*

We can now use this in a definition of a noncommutative sheaf [2].

Definition 3.5 *[2] Given (E, ∇) a left A -module with covariant derivative and zero curvature, define $H^*(A; E, \nabla)$ to be the cohomology of the cochain complex*

$$E \xrightarrow{\nabla} \Omega^1 A \otimes_A E \xrightarrow{\nabla^{[1]}} \Omega^2 A \otimes_A E \xrightarrow{\nabla^{[2]}} \dots\dots\dots$$

Note that $H^0(E, \nabla) = \{e \in E : \nabla e = 0\}$, the flat sections of E . We will often write $H^(A; E)$ where there is no danger of confusing the covariant derivative.*

We will take this opportunity to make a couple of well known statements about modules over algebras which we will use, as it may make the reading later easier for non-experts (see e.g. [3]).

Definition 3.6 A right A -module E is flat if every short exact sequence of left A -modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

gives another short exact sequence

$$0 \longrightarrow E \otimes_A L \longrightarrow E \otimes_A M \longrightarrow E \otimes_A N \longrightarrow 0.$$

Similarly, a left A -module F is called flat if $- \otimes_A F$ preserves exactness of short sequences of right modules.

Lemma 3.7 Given two short exact sequences of modules (left or right),

$$\begin{aligned} 0 \longrightarrow U &\xrightarrow{t} V \xrightarrow{f} W \longrightarrow 0, \\ 0 \longrightarrow U &\xrightarrow{t} V \xrightarrow{g} X \longrightarrow 0, \end{aligned}$$

there is an isomorphism $h : W \longrightarrow X$ given by $h(w) = g(v)$, where $f(v) = w$.

4 Differential fibrations and the Serre spectral sequence

4.1 A simple differential fibration

The reader may take this section as a justification of why the definition of a noncommutative differential fibration which we will give in Definition 4.1 is reasonable. Take a trivial fibration $\pi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ given by

$$(x_1, \dots, x_n, y_1, \dots, y_m) \longmapsto (x_1, \dots, x_n) .$$

Here the base space is $B = \mathbb{R}^n$, the fiber is \mathbb{R}^m , and the total space is $E = \mathbb{R}^{n+m}$. We can write a basis for the differential forms on the total space, putting the B terms (the dx_i) first. A form of degree p in the base and q in the fiber (total degree $p + q$) is

$$dx_{i_1} \wedge \dots \wedge dx_{i_p} \wedge dy_{j_1} \wedge \dots \wedge dy_{j_q} ,$$

for example $dx_2 \wedge dx_4 \wedge dy_1 \wedge dy_7 \wedge dy_9$. If we have the projection map $\pi : E \longrightarrow B$, we can write our example form as

$$\alpha = \pi^*(dx_2 \wedge dx_4) \wedge (dy_1 \wedge dy_7 \wedge dy_9)$$

so we have a form in $\pi^*\Omega^2 B \wedge \Omega^3 E$. Another element of $\pi^*\Omega^2 B \wedge \Omega^3 E$ might be

$$\beta = \pi^*(dx_2 \wedge dx_4) \wedge (dx_3 \wedge dy_1 \wedge dy_7).$$

Note, we now just look at $\Omega^3 E$, not the forms in the fiber direction, as in the noncommutative case we will not know (at least in the begining) what the fiber is. We need to describe the forms on the fiber space more indirectly. Now look at the vector space quotient

$$\frac{\pi^* \Omega^2 B \wedge \Omega^3 E}{\pi^* \Omega^3 B \wedge \Omega^2 E} . \quad (2)$$

Here β is also an element of the bottom line of (2), as we could write

$$\beta = \pi^*(dx_2 \wedge dx_4 \wedge dx_3) \wedge (dy_1 \wedge dy_7)$$

so, denoting the quotient by square brackets, $[\beta] = 0$. On the other hand, α is not in the bottom line of (2), so $[\alpha] \neq 0$. We can now use

$$\frac{\pi^* \Omega^p B \wedge \Omega^q E}{\pi^* \Omega^{p+1} B \wedge \Omega^{q-1} E}$$

to denote the forms on the total space which are of degree p in the base and degree q in the fiber, without explicitly having any coordinates for the fiber. This is just the idea of a noncommutative differential fibration.

4.2 Noncommutative differential fibrations

In Subsection 4.1 we had a topological fibration $\pi : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n$. For algebras, we will reverse the arrows, and look at $\iota : B \rightarrow A$, where B is the ‘base algebra’ and A is the ‘total algebra’.

Suppose that both A and B have differential calculi, and that the algebra map $\iota : B \rightarrow A$ is differentiable. This means that $\iota : B \rightarrow A$ extends to a map of differential graded algebras $\iota_* : \Omega^* B \rightarrow \Omega^* A$, and in particular that $d \iota_* = \iota_* d$ and $\iota_* \wedge = \wedge (\iota_* \otimes \iota_*)$. Now we set

$$D_{p,q} = \iota_* \Omega^p B \wedge \Omega^q A \quad \text{and} \quad N_{p,q} = \frac{D_{p,q}}{D_{p+1,q-1}} , \quad N_{p,0} = \iota_* \Omega^p B . A . \quad (3)$$

Now we can finally define a differential fibration, remembering that we use $[]$ to denote equivalence class in the quotient in (3):

Definition 4.1 $\iota : B \rightarrow A$ is a differential fibration if the map

$$\xi \otimes [x] \longrightarrow [\iota_* \xi \wedge x]$$

gives an isomorphism from $\Omega^p B \otimes_B N_{0,q}$ to $N_{p,q}$ for all $p, q \geq 0$.

Example 4.2 (See section 8.5 of [2].) Given the left covariant calculus on the quantum group $SU_q(2)$ given by Woronowicz [16], the corresponding differential calculus on the quantum sphere S_q^2 gives a differential fibration

$$\iota : S_q^2 \longrightarrow SU_q(2) .$$

Here the algebra SU_q^2 is the invariants of $SU_q(2)$ under a circle action, and ι is just the inclusion.

We will give another example in Section 6. Now we have the following version of the Serre spectral sequence from [2].

Theorem 4.3 Suppose that $\iota : B \rightarrow X$ is a differential fibration. Then there is a spectral sequence converging to $H_{dR}^*(A)$ with

$$E_2^{p,q} \cong H^p(B; H^q(N_{0,*}), \nabla) .$$

Here ∇ is a zero curvature covariant derivative on the left B -modules $N_{0,n}$, whose construction we will not go further into, as we are about to something more general.

5 The noncommutative Leray spectral sequence

5.1 A filtration of a cochain complex

We suppose that E is a left A module, with a left covariant derivative

$$\nabla : E \longrightarrow \Omega^1 A \otimes_A E$$

and that this covariant derivative is flat, i.e. that its curvature vanishes. Then $\nabla^{[n]} : \Omega^n A \otimes_A E \longrightarrow \Omega^{n+1} A \otimes_A E$ is a cochain complex (see definition 3.5). Suppose that $\iota_* : \Omega^* B \longrightarrow \Omega^* A$ is a map of differential graded algebras. We define a filtration of $\Omega^n A \otimes_A E$ by

$$F^m(\Omega^n A \otimes_A E) = \begin{cases} \iota_* \Omega^m B \wedge \Omega^{n-m} A \otimes_A E & 0 \leq m \leq n; \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Proposition 5.1 The filtration in (4) satisfies the conditions of remark 2.1.

Proof: First $F^0(\Omega^n A \otimes_A E) = \iota_* \Omega^0 B \wedge \Omega^n A \otimes_A E$, but $1 \in \iota_* \Omega^0 B = \iota_* B$, so $F^0(\Omega^n A \otimes_A E) = \Omega^n A \otimes_A E$.

To show it is decreasing, (using condition (5) from definition 3.1)

$$\begin{aligned}
F^{m+1}(\Omega^n A \otimes_A E) &= \iota_* \Omega^{m+1} B \wedge \Omega^{n-m-1} A \otimes_A E \\
&= \iota_* \Omega^m B \wedge (\iota_* \Omega^1 B \wedge \Omega^{n-m-1} A) \otimes_A E \\
&\subset \iota_* \Omega^m B \wedge \Omega^{n-m} A \otimes_A E \\
&\subset F^m(\Omega^n A \otimes_A E) .
\end{aligned}$$

To show that the filtration is preserved by d , take $\iota_* \xi \wedge \eta \otimes e \in F^m(\Omega^n A \otimes_A E)$ where $\xi \in \Omega^m B$, and $\eta \in \Omega^{n-m} A$. Then

$$d(\iota_* \xi \wedge \eta \otimes e) = \iota_* d\xi \wedge \eta \otimes e + (-1)^m \iota_* \xi \wedge d\eta \otimes e + (-1)^n \iota_* \xi \wedge \eta \wedge \nabla e$$

This is in $F^m C$, as the first term is in $F^{m+1} C \subset F^m C$, and the other two are in $F^m C$. \square

Now we have a spectral sequence which converges to $H_{dR}^*(A; E)$. All we have to do is to find the first and second pages of the spectral sequence, though this is quite lengthy.

5.2 The first page of the spectral sequence

From section 2.2, to use the filtration in section 5.1 we need to work with

$$M_{p,q} = \frac{F^p C^{p+q}}{F^{p+1} C^{p+q}} = \frac{\iota_* \Omega^p B \wedge \Omega^q A \otimes_A E}{\iota_* \Omega^{p+1} B \wedge \Omega^{q-1} A \otimes_A E} \quad (5)$$

Then we look, for p fixed (following (1)), at the sequence

$$\cdots M_{p,q-1} \xrightarrow{d} M_{p,q} \xrightarrow{d} M_{p,q+1} \xrightarrow{d} \cdots \quad (6)$$

as the cohomology of this sequence gives the first page of the spectral sequence. Denote the quotient in $M_{p,q}$ by $[\]_{p,q}$, so if $x \in \iota_* \Omega^p B \wedge \Omega^q A \otimes_A E$, then $[x]_{p,q} \in M_{p,q}$. Then we have a map of left B modules

$$\Omega^p B \otimes_B M_{0,q} \longrightarrow M_{p,q} , \quad \xi \otimes [y]_{0,q} \longmapsto [\iota_* \xi \wedge y]_{p,q} .$$

Here $y \in \Omega^q A \otimes_A E$ and the left action of $b \in B$ on y is $\iota(b)y$.

Proposition 5.2 *If E is flat as a left A module, then $N_{p,q} \otimes_A E \cong M_{p,q}$ with isomorphism $[z] \otimes e \longmapsto [z \otimes e]_{p,q}$.*

Proof: We have, by definition, a short exact sequence using notation from (3), where inc is inclusion and $[\]$ is quotient,

$$0 \longrightarrow D_{p+1,q-1} \xrightarrow{\text{inc}} D_{p,q} \xrightarrow{[\]} N_{p,q} \longrightarrow 0.$$

As E is flat, we get another short exact sequence,

$$0 \longrightarrow D_{p+1,q-1} \otimes_A E \xrightarrow{\text{inc} \otimes \text{id}} D_{p,q} \otimes_A E \xrightarrow{[\] \otimes \text{id}} N_{p,q} \otimes_A E \longrightarrow 0$$

but by definition we also have

$$0 \longrightarrow D_{p+1,q-1} \otimes_A E \xrightarrow{\text{inc} \otimes \text{id}} D_{p,q} \otimes_A E \xrightarrow{[\]_{p,q}} M_{p,q} \longrightarrow 0.$$

and the result follows from Lemma 3.7. \square

Proposition 5.3 *If E is a flat left A module, and $\iota : B \longrightarrow A$ is a fibering in the sense of definition 4.1, then*

$$\Omega^p B \otimes_B N_{0,q} \otimes_A E \cong M_{p,q}$$

via the map

$$\xi \otimes [x] \otimes e \longmapsto [\iota_* \xi \wedge x \otimes e]_{p,q}.$$

Proof: Definition 4.1 gives an isomorphism

$$\Omega^p B \otimes_B N_{0,q} \longrightarrow N_{p,q}$$

by $\xi \otimes [x] \longmapsto [\iota_* \xi \wedge x]$. Now use Proposition 5.2. \square

We now return to the problem of calculating the cohomology of the sequence (6). Take $\xi \otimes [x] \otimes e \in \Omega^p B \otimes_B N_{0,q} \otimes_A E$ (for $x \in \Omega^q A$) which maps to $[\iota_* \xi \wedge x \otimes e] \in M_{p,q}$, and apply the differential $\nabla^{[p+q]}$ to it to get

$$\begin{aligned} & d(\iota_* \xi \wedge x) \otimes e + (-1)^{p+q} \iota_* \xi \wedge x \wedge \nabla e \\ &= \iota_* d\xi \wedge x \otimes e + (-1)^p \iota_* \xi \wedge dx \otimes e + (-1)^{p+q} \iota_* \xi \wedge x \wedge \nabla e. \end{aligned} \quad (7)$$

But $d\xi \in \Omega^{p+1} B$, and

$$M_{p,q+1} = \frac{\iota_* \Omega^p B \wedge \Omega^{q+1} A \otimes_A E}{\iota_* \Omega^{p+1} B \wedge \Omega^q A \otimes_A E},$$

so the first term vanishes on applying $[\]_{p,q+1}$. Then

$$d[\iota_* \xi \wedge x \otimes e]_{p,q} = (-1)^p [\iota_* \xi \wedge (dx \otimes e + (-1)^q x \wedge \nabla e)]_{p,q+1} \quad (8)$$

Then, using Proposition 5.3, we have an isomorphism

$$\Omega^p B \otimes_B M_{0,q} \cong M_{p,q} , \quad \xi \otimes [y]_{0,q} \longmapsto [\iota_* \xi \wedge y]_{p,q} , \quad (9)$$

and using this isomorphism, d on $M_{p,q}$ can be written as (see 8)

$$d(\xi \otimes [y]_{0,q}) = (-1)^p \xi \otimes [\nabla^{[q]} y]_{0,q+1} \quad (10)$$

where $y \in \Omega^q A \otimes_A E$. From (10) we see that we should study $[\nabla^{[q]}] : M_{0,q} \longrightarrow M_{0,q+1}$, defined by $[y]_{0,q} \longmapsto [\nabla^{[q]} y]_{0,q+1}$.

Proposition 5.4 $[\nabla^{[q]}] : M_{0,q} \longrightarrow M_{0,q+1}$ is a left B module map. The module structure is $b \cdot [\eta \otimes e] = [i(b)\eta \otimes e]$, for $b \in B$ and $\eta \otimes e \in \Omega^q A \otimes_A E$.

Proof: First,

$$\begin{aligned} [\nabla^{[q]}](b \cdot [\eta \otimes e]_{0,q}) &= [d(i(b)\eta) \otimes e + (-1)^q i(b)\eta \wedge \nabla e]_{0,q+1} \\ &= [\iota_*(db) \wedge \eta \otimes e + i(b) \cdot d\eta \otimes e + (-1)^q i(b)\eta \wedge \nabla e]_{0,q+1} \end{aligned}$$

Now

$$\iota_*(db) \wedge \eta \otimes e \in \iota_* \Omega^1 B \wedge \Omega^q A \otimes_A E$$

so $[\iota_*(db) \wedge \eta \otimes e]_{0,q+1} = 0$ in $M_{0,q+1}$. Then

$$\begin{aligned} [\nabla^{[q]}](b \cdot [\eta \otimes e]_{0,q}) &= [i(b) \cdot d\eta \otimes e + (-1)^q i(b)\eta \wedge \nabla e]_{0,q+1} \\ &= b \cdot [d\eta \otimes e + (-1)^q \eta \wedge \nabla e]_{0,q+1}. \quad \square \end{aligned}$$

Proposition 5.5 If $\Omega^p B$ is flat as a right B module, the cohomology of the cochain complex

$$\cdots M_{p,q-1} \xrightarrow{d} M_{p,q} \xrightarrow{d} M_{p,q+1} \xrightarrow{d} \cdots$$

is given by $\Omega^p B \otimes_B \hat{H}_q$, where \hat{H}_q is defined as the cohomology of the cochain complex

$$\cdots \xrightarrow{[\nabla^{[q-1]}]} M_{0,q} \xrightarrow{[\nabla^{[q]}]} M_{0,q+1} \xrightarrow{[\nabla^{[q+1]}]} \cdots$$

If we write $\langle \cdot \rangle_{p,q}$ for the equivalence class in the cohomology of $M_{p,q}$, this isomorphism is given by, for $\xi \in \Omega^p B$ and $x \in \Omega^q A \otimes_A E$,

$$\langle \iota_* \xi \wedge x \rangle_{p,q} \longrightarrow \xi \otimes \langle x \rangle_{0,q} . \quad (11)$$

Proof: To calculate the cohomology, we need to find $Z_{p,q} = \text{im } d : M_{p,q-1} \rightarrow M_{p,q}$ and $K_{p,q} = \ker d : M_{p,q} \rightarrow M_{p,q+1}$. As we know from Proposition 5.4 that $d = [\nabla^{[q]}] : M_{0,q} \rightarrow M_{0,q+1}$ is a left B module map, we have an exact sequence of left B modules, where the first map is inclusion,

$$0 \longrightarrow K_{0,q} \xrightarrow{\text{inc}} M_{0,q} \xrightarrow{d} Z_{0,q+1} \longrightarrow 0 . \quad (12)$$

Since $\Omega^p B$ is flat as a right B module, we have another exact sequence,

$$0 \longrightarrow \Omega^p B \otimes_B K_{0,q} \xrightarrow{\text{id} \otimes \text{inc}} \Omega^p B \otimes_B M_{0,q} \xrightarrow{\text{id} \otimes d} \Omega^p B \otimes_B Z_{0,q+1} \longrightarrow 0 . \quad (13)$$

Now refer to the isomorphism given in (9), and then by (10) the last map $\text{id} \otimes d$ is $(-1)^p d$ on $M_{p,q}$, so $Z_{p,q} = \Omega^p B \otimes_B Z_{0,q}$ and $K_{p,q} = \Omega^p B \otimes_B K_{0,q}$.

From the definition of \hat{H}_q we have another short exact sequence,

$$0 \longrightarrow Z_{0,q} \xrightarrow{\text{inc}} K_{0,q} \longrightarrow \hat{H}_q \longrightarrow 0 ,$$

and applying $\Omega^p B \otimes_B$ gives, as $\Omega^p B$ is flat as a right B module,

$$0 \longrightarrow \Omega^p B \otimes_B Z_{0,q} \xrightarrow{\text{id} \otimes \text{inc}} \Omega^p B \otimes_B K_{0,q} \longrightarrow \Omega^p B \otimes_B \hat{H}_q \longrightarrow 0 . \quad (14)$$

We deduce that the cohomology of $M_{p,q}$ is isomorphic to $\Omega^p B \otimes_B \hat{H}_q$. \square

5.3 The second page of the spectral sequence

Now we move to the second page of the spectral sequence, in which we take the cohomology of the previous cohomology, i.e. the cohomology of

$$d : \text{cohomology}(M_{p,q}) \longrightarrow \text{cohomology}(M_{p+1,q}).$$

By the isomorphism discussed in Proposition 5.5, we can view this as

$$d : \Omega^p B \otimes_B \hat{H}_q \longrightarrow \Omega^{p+1} B \otimes_B \hat{H}_q . \quad (15)$$

Proposition 5.6 *The differential d gives a left covariant derivative*

$$\nabla_q : \hat{H}_q \longrightarrow \Omega^1 B \otimes_B \hat{H}_q .$$

If $\langle \xi \otimes e \rangle_{0,q} \in \hat{H}_q$, this is given by using the isomorphism (11) as

$$\langle \xi \otimes e \rangle_{0,q} \longmapsto \eta \otimes \langle \omega \otimes f \rangle_{0,q} ,$$

where

$$d\xi \otimes e + (-1)^q \xi \wedge \nabla e = \iota_* \eta \wedge \omega \otimes f \in \iota_* \Omega^1 B \wedge \Omega^q A \otimes_A E .$$

Proof: Take $\langle x \rangle_{0,q} \in \hat{H}_q$, where $x \in K_{0,q} = \ker d : M_{0,q} \rightarrow M_{0,q+1}$, and suppose $x = \xi \otimes e$, where $\xi \in \Omega^q A$ and $e \in E$ (summation implicit). As $x \in K_{0,q}$ we have

$$[dx]_{0,q+1} = [d\xi \otimes e + (-1)^q \xi \wedge \nabla e]_{0,q+1} = 0$$

in $M_{0,q+1}$, so

$$d\xi \otimes e + (-1)^q \xi \wedge \nabla e \in \iota_* \Omega^1 B \wedge \Omega^q A \otimes_A E.$$

We write (summation implicit), for $\eta \in \Omega^1 B$, $\omega \in \Omega^1 A$ and $f \in E$,

$$d\xi \otimes e + (-1)^q \xi \wedge \nabla e = \iota_* \eta \wedge \omega \otimes f. \quad (16)$$

Under the isomorphism (9), this corresponds to $\eta \otimes [\omega \otimes f]_q \in \Omega^1 B \otimes_B M_{0,q}$. As the curvature of E vanishes, we have from applying $\nabla^{[q+1]}$ to (16),

$$\iota_* d\eta \wedge \omega \otimes f - \iota_* \eta \wedge d\omega \otimes f + (-1)^{q+1} \iota_* \eta \wedge \omega \wedge \nabla f = 0. \quad (17)$$

We take this as an element of $M_{1,q+1}$, so we apply $[\]_{1,q+1}$ to (17). Then as the denominator of $M_{1,q+1}$ is $\iota_* \Omega^2 B \wedge \Omega^q A \otimes_A E$, we see that the first term of (17) vanishes on taking the quotient, giving

$$-[\iota_* \eta \wedge (d\omega \otimes f + (-1)^q \omega \wedge \nabla f)]_{1,q+1} = 0.$$

Under the isomorphism (9) this corresponds to

$$-\eta \otimes_B [d\omega \otimes f + (-1)^q \omega \wedge \nabla f]_{0,q+1} = 0. \quad (18)$$

This means that

$$\eta \otimes [\omega \otimes f]_{0,q} \in \Omega^1 B \otimes_B M_{0,q}$$

is in the kernel of the map $\text{id} \otimes d$ in (13), and as (13) is an exact sequence we have

$$\eta \otimes [\omega \otimes f]_{0,q} \in \Omega^1 B \otimes_B K_{0,q},$$

so we can see take the cohomology class to get

$$\eta \otimes \langle \omega \otimes f \rangle_{0,q} \in \Omega^1 B \otimes_B \hat{H}_q.$$

This completes showing that ∇_q exists, but we also need to show that it is a left covariant derivative. For $b \in B$, we calculate $\nabla_q(b.\xi \otimes e)$ to get

$$d(b.\xi) \otimes e + (-1)^q b.\xi \wedge \nabla e = db \wedge \xi \otimes e + b.(d\xi \otimes e + (-1)^q \xi \wedge \nabla e),$$

so we get

$$\nabla_q \langle b.\xi \otimes e \rangle_{0,q} = db \otimes \langle \xi \otimes e \rangle_{0,q} + b.\nabla_q \langle \xi \otimes e \rangle_{0,q}. \quad \square$$

Proposition 5.7 *The curvature R_q of the covariant derivative ∇_q in Proposition 5.6 is zero.*

Proof: Using the notation of Proposition 5.6, equation (16)

$$\nabla_q \langle \xi \otimes e \rangle_{0,q} = \eta \otimes \langle \omega \otimes f \rangle_{0,q}.$$

If we apply $\nabla_q^{[1]}$ (see Definition 3.5) to this, we get

$$R_q \langle \xi \otimes e \rangle_{0,q} = d\eta \otimes \langle \omega \otimes f \rangle_{0,q} - \eta \wedge \nabla_q \langle \omega \otimes f \rangle_{0,q}. \quad (19)$$

To find $\nabla_q \langle \omega \otimes f \rangle_{0,q}$, referring to the proof of Proposition 5.6, formula (18), we have

$$\eta \otimes_B (d\omega \otimes f + (-1)^q \omega \wedge \nabla f) \in \Omega^1 B \otimes_B (\iota_* \Omega^1 B \wedge \Omega^q A \otimes_A E).$$

This comes from tensoring the exact sequence

$$0 \longrightarrow \iota_* \Omega^1 B \wedge \Omega^q A \otimes_A E \longrightarrow \Omega^{q+1} A \otimes_A E \xrightarrow{[\]_{0,q+1}} M_{0,q+1} \longrightarrow 0$$

on the left by $\Omega^1 B$, and using that $\Omega^1 B$ is a flat right module. Now write (summation implicit),

$$\eta \otimes (d\omega \otimes f + (-1)^q \omega \wedge \nabla f) = \eta' \otimes (\iota_* \kappa \wedge \zeta \otimes g) \quad (20)$$

for $\eta', \kappa \in \Omega^1 B$, $\zeta \in \Omega^q A$ and $g \in E$. Then, from Proposition 5.6,

$$\eta \wedge \nabla_q \langle \omega \otimes f \rangle_{0,q} = \eta' \wedge \kappa \otimes \langle \zeta \otimes g \rangle_{0,q}$$

so from (19),

$$R_q \langle \xi \otimes e \rangle_{0,q} = d\eta \otimes \langle \omega \otimes f \rangle_{0,q} - \eta' \wedge \kappa \otimes \langle \zeta \otimes g \rangle_{0,q}. \quad (21)$$

Now (20) implies that

$$\iota_* \eta \wedge (d\omega \otimes f + (-1)^q \omega \wedge \nabla f) = \iota_* \eta' \wedge \iota_* \kappa \wedge \zeta \otimes g,$$

and substituting this into (17) gives

$$\iota_* d\eta \wedge \omega \otimes f - \iota_* \eta' \wedge \iota_* \kappa \wedge \zeta \otimes g = 0,$$

so on taking equivalence classes in $M_{2,q}$ we find, using the isomorphism (9),

$$d\eta \otimes [\omega \otimes f]_{0,q} - \eta' \wedge \kappa \otimes [\zeta \otimes g]_{0,q} = 0,$$

and this shows that $R_q = 0$ by (21). \square

Theorem 5.8 *Given:*

- 1) a map $\iota : B \longrightarrow A$ which is a differential fibration (see definition 4.1),
 - 2) a flat left A module E , with a zero-curvature left covariant derivative $\nabla_E : E \rightarrow \Omega^1 A \otimes_A E$,
 - 3) each $\Omega^p B$ is flat as a right B module,
- then there is a spectral sequence converging to $H^*(A, E, \nabla_E)$ with second page $H^*(B, \hat{H}_q, \nabla_q)$ where \hat{H}_q is defined as the cohomology of the cochain complex

$$\cdots \xrightarrow{d} M_{0,q} \xrightarrow{d} M_{0,q+1} \xrightarrow{d} \cdots$$

where

$$\begin{aligned} M_{0,q} &= \frac{\Omega^q A \otimes_A E}{\iota_* \Omega^1 B \wedge \Omega^{q-1} A \otimes_A E}, \\ d[x \otimes e]_{0,q} &= [dx \otimes e + (-1)^q x \wedge \nabla_E e]_{0,q+1}. \end{aligned}$$

The zero curvature left covariant derivative $\nabla_q : \hat{H}_q \rightarrow \Omega^1 B \otimes_B \hat{H}_q$ is as defined in Proposition 5.6.

Proof: The first part of the proof is given in Proposition 5.5. Now we need to calculate the cohomology of

$$d : \Omega^p B \otimes_B \hat{H}_q \longrightarrow \Omega^{p+1} B \otimes_B \hat{H}_q$$

This is given for $\xi \otimes \langle \eta \otimes e \rangle_{0,q}$ (for $\xi \in \Omega^p B$, $\eta \in \Omega^q A$ and $e \in E$) as follows: this element corresponds to $\iota_* \xi \wedge \eta \otimes e$, and applying d to this gives

$$\iota_* d\xi \wedge \eta \otimes e + (-1)^p \iota_* \xi \wedge d\eta \otimes e + (-1)^{p+q} \iota_* \xi \wedge \eta \wedge \nabla e.$$

But we have calculated the effect of d on \hat{H}_q in Proposition 5.6, so we get

$$d(\xi \otimes \langle \eta \otimes e \rangle_{0,q}) = d\xi \otimes \langle \eta \otimes e \rangle_{0,q} + (-1)^p \xi \wedge \nabla_q \langle \eta \otimes e \rangle_{0,q}.$$

The covariant derivative ∇_q has zero curvature by Proposition 5.7. \square

6 Example: A fibration with fiber the noncommutative torus

As discussed in the Introduction, the idea for this example came from [9, 10].

6.1 The Heisenberg group

The Heisenberg group H is defined to be following subgroup of $M_3(\mathbb{Z})$ under multiplication.

$$\left\{ \begin{pmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} : n, m, k \in \mathbb{Z} \right\}$$

We can take generators u, v, w for the group, where w is central and there is one more relation $uv = wvu$. These generators correspond to the matrices

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

There is an isomorphism $\theta : H \longrightarrow H$, for every matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

given by $\theta(u) = u^a v^b$, $\theta(v) = u^c v^d$, $\theta(w) = w$. The group algebra $\mathbb{C}H$ of H can be made into a star algebra by setting $x^* = x^{-1}$ for all $x \in \{u, v, w\}$.

6.2 A differential calculus on the Heisenberg group

There is a differential calculus on the group algebra $\mathbb{C}H$ of H . It is bico-variant, as set down by Woronowicz in [16].

For a generator $x \in \{u, v, w\}$, we write $e^x = x^{-1}.dx$, a left invariant element of $\Omega^1\mathbb{C}H$. We suppose that $\Omega^1\mathbb{C}H$ is free as left $\mathbb{C}H$ module, with generators $\{e^u, e^v, e^w\}$. This means that every element of $\Omega^1\mathbb{C}H$ can be written uniquely as $a^u.e^u + a^v.e^v + a^w.e^w$, for $a^u, a^v, a^w \in \mathbb{C}H$. We have the following relations on $\Omega^1\mathbb{C}H$, for all $x \in \{u, v, w\}$:

$$x.e^x = e^x.x$$

$$x.e^w = e^w.x$$

$$w.e^x = e^x.w$$

$$u^{-n}.e^v.u^n = e^v - \frac{n}{2}e^w$$

$$v^{-n}.e^u.v^n = e^u + \frac{n}{2}e^w$$

Further the map θ in subsection 6.1 extends to a map of 1-forms given by

$$\theta(e^w) = e^w$$

$$\theta(e^u) = a.e^u + b.e^v + \frac{ab}{2}.e^w$$

$$\theta(e^v) = c.e^u + d.e^v + \frac{cd}{2}.e^w$$

Checking the braiding given by Woronowicz shows that, for $x, y \in \{u, v, w\}$,

$$de^x = 0$$

$$e^x \wedge e^y = -e^y \wedge e^x .$$

The star operation extends to the differential calculus, with $(e^x)^* = -e^x$.

6.3 The differential fibration

If we take z to be the identity function $:S^1 \rightarrow \mathbb{C}$, the map sending z^n to w^n gives an algebra map $\iota : C(S^1) \rightarrow \mathbb{C}H$. It is also a star algebra map, with the usual star structure $z^* = z^{-1}$ on $C(S^1)$.

The differential structure of the ‘fiber algebra’ F is

$$\Omega^n F = \frac{\Omega^n \mathbb{C}H}{\iota_* \Omega^1 C(S^1) \wedge \Omega^{n-1} \mathbb{C}H} , \quad (22)$$

i.e. we put $dw = 0$ in $\Omega^n F$ (i.e. put $e^w = 0$). This is because in (22) we divide by everything of the form $e^w \wedge \xi$. To see that this gives a fibration, we note that a linear basis for the left invariant n-forms is as follows:

$$\Omega^1 A: \quad e^u, e^v, e^w$$

$$\Omega^2 A: \quad e^u \wedge e^v, e^w \wedge e^u, e^w \wedge e^v$$

$$\Omega^3 A: \quad e^v \wedge e^u \wedge e^w$$

Then the $N_{n,m}$ (see (3)) are, where $\langle \dots \rangle$ denotes the module generated by, and all others are zero:

$$N_{0,0} = 1, \quad N_{1,0} = \langle e^w \rangle, \quad N_{m,0} = 0, \quad m > 1$$

$$N_{0,1} = \frac{\langle e^u, e^v, e^w \rangle}{\langle e^w \rangle} = \langle e^u, e^v \rangle$$

$$N_{0,2} = \frac{\langle e^u \wedge e^v, e^w \wedge e^u, e^w \wedge e^v \rangle}{\langle e^w \wedge e^u, e^w \wedge e^v \rangle} = \langle e^u \wedge e^v \rangle$$

$$N_{0,3} = \frac{\langle e^w \wedge e^u \wedge e^v \rangle}{\langle e^w \wedge e^u \wedge e^v \rangle} = 0$$

$$N_{0,n} = 0 \quad n \geq 4$$

$$N_{1,1} = \frac{e^w \wedge \langle e^u, e^v \rangle}{\langle 0 \rangle} = \langle e^w \wedge e^u, e^w \wedge e^v \rangle$$

$$N_{1,2} = \frac{e^w \wedge \langle e^u \wedge e^v, e^w \wedge e^u, e^w \wedge e^v \rangle}{\langle 0 \rangle} = \langle e^w \wedge e^u \wedge e^v \rangle$$

Then the following map is one-to-one and onto,

$$\Omega^1 C(S^1) \otimes_{C(S^1)} N_{0,n} \longrightarrow N_{1,n}$$

giving a differential fibration in the sense of Definition 4.1.

As was done in [9], we note that this map does have a fiber in quite a classical sense. The algebra $C(S^1)$ is commutative, and if we take $q \in S^1$, the fiber algebra corresponding to q is given by substituting $w \mapsto q$ in the algebra relations. We get unitary generators u, v and a relation $uv = qvu$ for a complex number q of norm 1. But this is exactly the noncommutative torus \mathbb{T}_q^2 . The map θ on the total algebra $\mathbb{C}H$ is the identity on the base algebra $C(S^1)$, so it acts on each fiber.

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